Arterial Flow Modelling

MEC434 - Biomechanics of the Cardiovascular System

Alessandro Melis
a.melis@sheffield.ac.uk

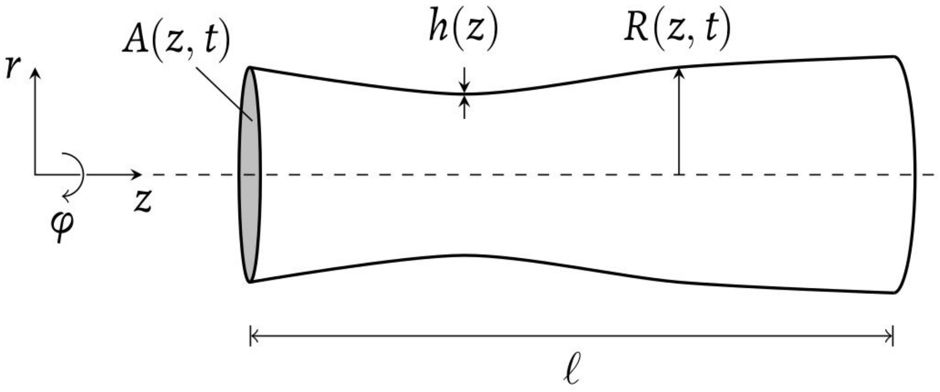
Dept. Mechanical Engineering
INSIGNEO Institute for *in silico* medicine
PLB E09

Outline

- Single vessel model assumptions
- Conservation laws
- Navier-Stokes equations 3D - 1D
- Velocity profile
- Constitutive equation
- Three element windkessel
- Coupling 1D and 0D
- Systemic circulation network
- Cerebral circulation

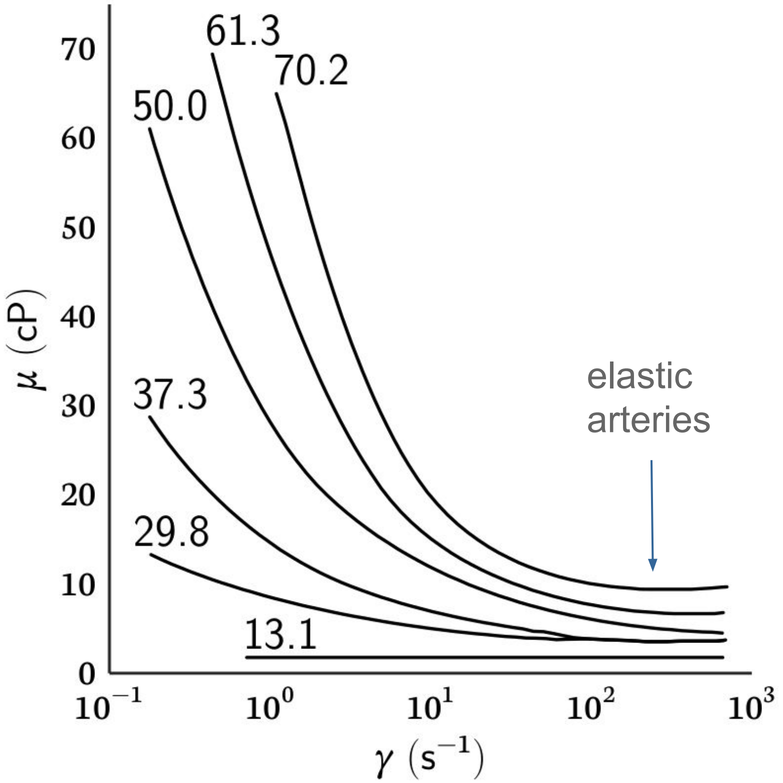
Single vessel

- Geometry
  - Straight: no bends
  - Axisymmetric: circular cross-sectional area
  - Narrow: $l >> R$
  - Thin: $h < R$
- Mechanical properties
  - Linear elastic walls
  - Only small radial displacement allowed



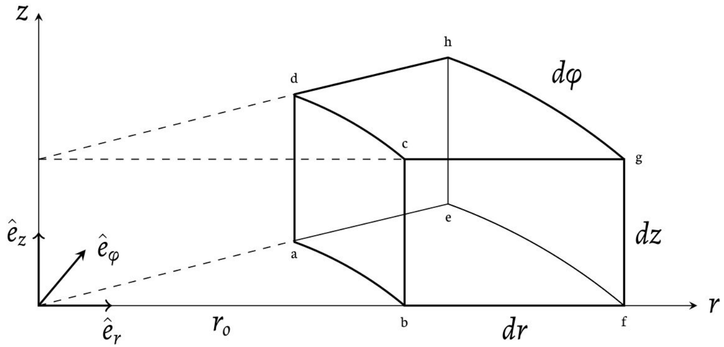
Blood properties

- Constant density: incompressible flow
- Newtonian fluid: constant dynamic viscosity w.r.t. shear rate
- Hematocrit: volume percentage (vol%) of red blood cells in blood.



Conservation laws

- Conservation of mass
- Conservation of momentum



density

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = \mathbf{0}$$

dynamic viscosity

body forces

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\, \mathbf{v} = -\frac{\mathbf{1}}{\rho} \nabla P + \frac{\mu}{\rho} \Delta \mathbf{v} + \mathbf{F}$$

time

velocity vector field

pressure

Assumptions

- No body forces: gravitational effect is negligible as the flow is driven by the heart contraction
- Incompressible flow: blood velocity <<< pulse wave velocity

$$\cancel{\frac{\partial \rho}{\partial t}} + \nabla \cdot (\rho \mathbf{v}) = \mathbf{0},$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\,\mathbf{v} = -\frac{\mathbf{1}}{\rho} \nabla P + \frac{\mu}{\rho} \Delta \mathbf{v} + \cancel{\mathbf{F}}$$

3D Navier-Stokes equations in cylindrical coordinates

- Newtonian fluid
- Axisymmetric flow: φ-wise terms neglected
- Integrate along *r*

$$\frac{\partial v_z}{\partial z} + \frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \cancel{\frac{1}{r}\frac{\partial v_\varphi}{\partial \varphi}} = 0,$$

$$\frac{\partial v_z}{\partial t} + v_z \frac{\partial v_z}{\partial z} + v_r \frac{\partial v_z}{\partial r} + \cancel{\frac{v_\varphi}{r}\frac{\partial v_z}{\partial \varphi}} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \frac{\mu}{\rho}\left[\frac{\partial^2 v_z}{\partial z^2} + \frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} + \cancel{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \varphi^2}}\right],$$

$$\frac{\partial v_r}{\partial t} + v_z \frac{\partial v_r}{\partial z} + v_r \frac{\partial v_r}{\partial r} + \cancel{\frac{v_\varphi}{r}\frac{\partial v_r}{\partial \varphi} - \frac{v_\varphi^2}{r}} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \frac{\mu}{\rho}\left[\frac{\partial^2 v_r}{\partial z^2} + \frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} + \cancel{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \varphi^2}} - \cancel{\frac{2}{r^2}\frac{\partial v_\varphi}{\partial \varphi}} - \frac{v_r}{r^2}\right],$$

$$\cancel{\frac{\partial v_\varphi}{\partial t} + v_z \frac{\partial v_\varphi}{\partial z} + v_r \frac{\partial v_\varphi}{\partial r} + \frac{v_\varphi}{r}\frac{\partial v_\varphi}{\partial \varphi} - \frac{v_r v_\varphi}{r} = -\frac{1}{\rho}\frac{\partial P}{\partial \varphi} + \frac{\mu}{\rho}\left[\frac{\partial^2 v_\varphi}{\partial z^2} + \frac{\partial^2 v_\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial v_\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_\varphi}{\partial \varphi^2} - \frac{2}{r^2}\frac{\partial v_r}{\partial \varphi} - \frac{v_\varphi}{r^2}\right].}$$

$$\downarrow$$

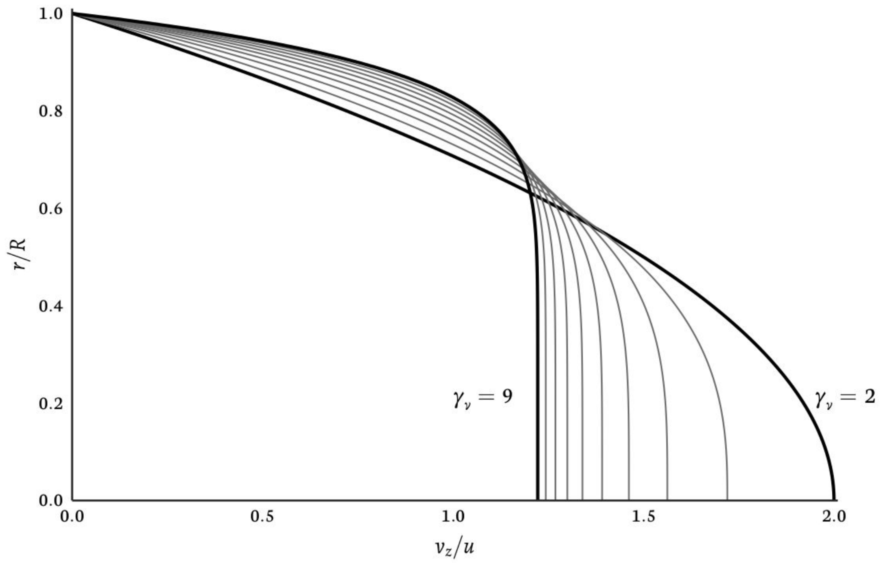
$$\frac{\partial (R^2 u)}{\partial t} + \frac{\partial}{\partial z}(\alpha R^2 u^2) + \frac{R^2}{\rho}\frac{\partial P}{\partial z} = 2\frac{\mu}{\rho} R \left.\frac{\partial v_z}{\partial r}\right|_R$$

Assumption on velocity profile

- The Coriolis' coefficient is related to γ_v
- The radial velocity profile can be expressed in function of γ_v
- $\gamma_v = 2$ parabolic profile
- $\gamma_v = 9$ plug-flow, flat profile

$$\alpha = \frac{\gamma_\nu + 2}{\gamma_\nu + 1}$$

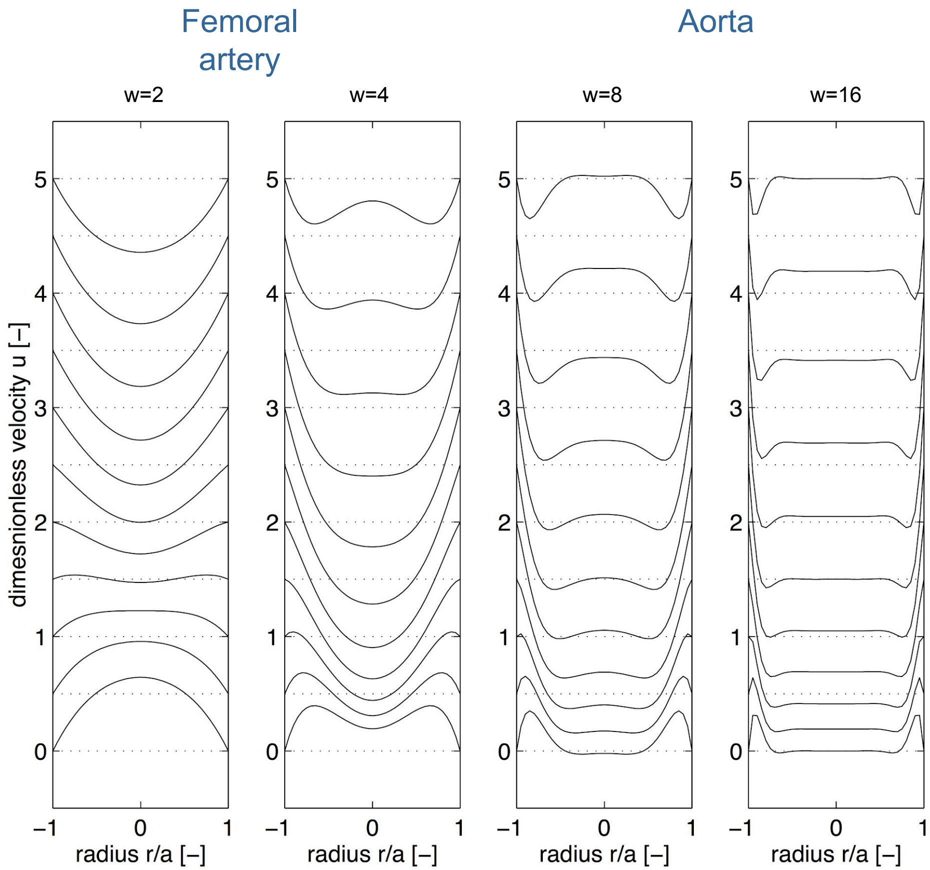
$$v_z = \frac{\gamma_\nu + 2}{\gamma_\nu} u \left[1 - \left(\frac{r}{R} \right)^{\gamma_\nu} \right]$$



Oscillatory flow

- Womersley number, *w*
- Flat profile: w = 16 (large elastic arteries)
- Parabolic profile: w = 2 (small arteries)

$$w = \ell \sqrt{\frac{2\pi\rho}{T_c \mu}}$$



Reduced equations

- 2 equations: continuity and momentum
- 2 dimensions: time (*t*) and *z* (longitudinal coordinate)
- 3 unknowns: *A, Q, P*

$$\begin{cases} \dfrac{\partial A}{\partial t} + \dfrac{\partial Q}{\partial z} = \mathbf{0}, \\[2ex] \dfrac{\partial Q}{\partial t} + \dfrac{\partial}{\partial z}\left(\alpha \dfrac{Q^2}{A}\right) + \dfrac{A}{\rho}\dfrac{\partial P}{\partial z} = -2\dfrac{\mu}{\rho}(\gamma_\nu + 2)\dfrac{Q}{A}. \end{cases}$$

Constitutive equation

- The tube circumferential stress can be described by means of Laplace's law.
- The strain is written as function of the lumen radius.
- By assuming negligible longitudinal displacement, the wall behaviour can be considered linearly elastic.
- The constitutive equation relates the internal pressure (*P*) to the tube cross-sectional area (*A*).

$$\sigma_\varphi = P\frac{R}{h_0}$$

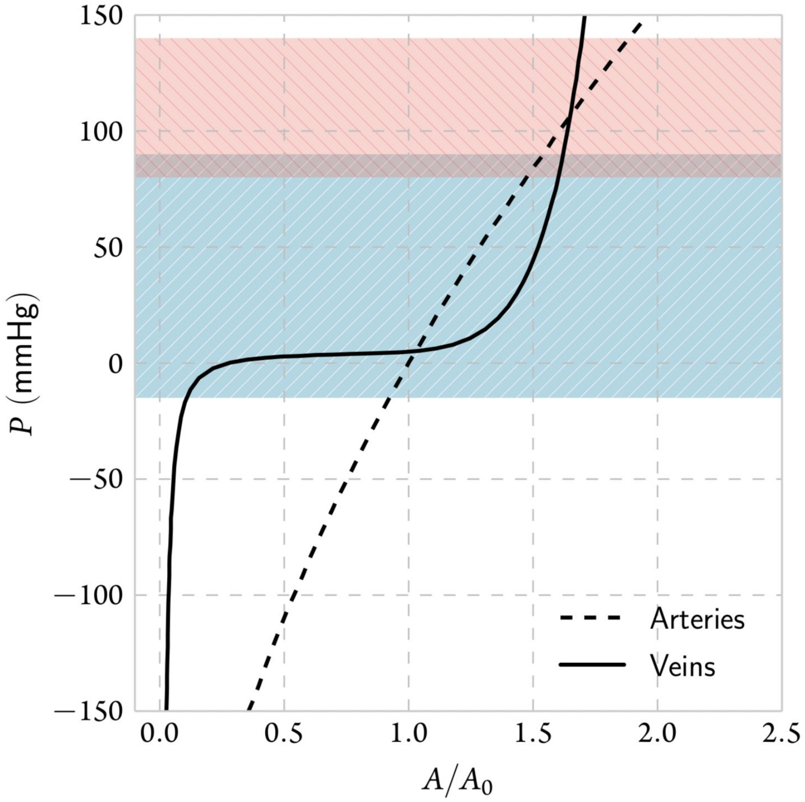
$$\varepsilon_\varphi = \frac{2\pi R - 2\pi R_0}{2\pi R_0}$$

$$\sigma_\varphi = \frac{E}{1-\nu^2}\varepsilon_\varphi$$

$$P(A) = \sqrt{\frac{\pi}{A_0}}\frac{Eh_0}{1-\nu^2}\left(\sqrt{\frac{A}{A_0}} - 1\right)$$

Arteries and veins

- Pressure as function of normalised cross-sectional area
- Shaded areas represent physiological domain
- Veins collapse in physiological conditions
- Arteries have a quasi-linear behaviour



1D model

- Non-linear hyperbolic system of PDEs
- Continuity equation
- Momentum equation with source term
- Constitutive relation in the transmural pressure equation
- Boundary conditions: inlet flow and outlet pressure

$$\begin{cases} \dfrac{\partial A}{\partial t} + \dfrac{\partial Q}{\partial z} = \mathbf{0}, \\[2ex] \dfrac{\partial Q}{\partial t} + \dfrac{\partial}{\partial z}\left(\alpha \dfrac{Q^2}{A}\right) + \dfrac{A}{\rho}\dfrac{\partial P}{\partial z} = -\mathbf{2}\dfrac{\mu}{\rho}(\gamma_\nu + \mathbf{2})\dfrac{Q}{A}, \\[2ex] P(A) = P_{ext} + \beta\left(\sqrt{\dfrac{A}{A_0}} - \mathbf{1}\right), \quad \beta = \sqrt{\dfrac{\pi}{A_0}}\dfrac{Eh_0}{\mathbf{1} - \nu^2}. \end{cases}$$

0D model

- Integrate also along *z*
- Express flow and pressure variable by means of averaged quantities
- Define *C, L,* and *R* constants

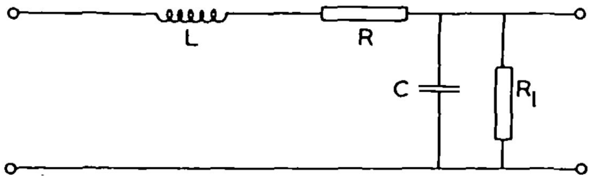
$$\begin{cases} \mathcal{C}\dfrac{d\overline{P}}{dt} + Q_\ell - Q_0 = 0, \\ \\ \mathcal{L}\dfrac{d\overline{Q}}{dt} + P_\ell - P_0 = -\mathcal{R}\overline{Q}, \end{cases}$$

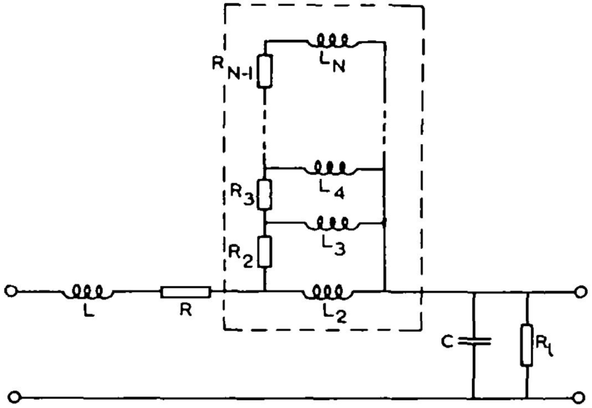
$$\overline{P} = \frac{1}{\ell}\int_0^\ell P dz, \quad \overline{Q} = \frac{1}{\ell}\int_0^\ell Q dz$$

$$\mathcal{C} = \frac{2A_0}{\beta\sqrt{\pi}}, \quad \mathcal{L} = \frac{\rho}{A_0}, \quad \mathcal{R} = \frac{\rho K_\nu}{A_0^2}$$

Electric circuit analogy

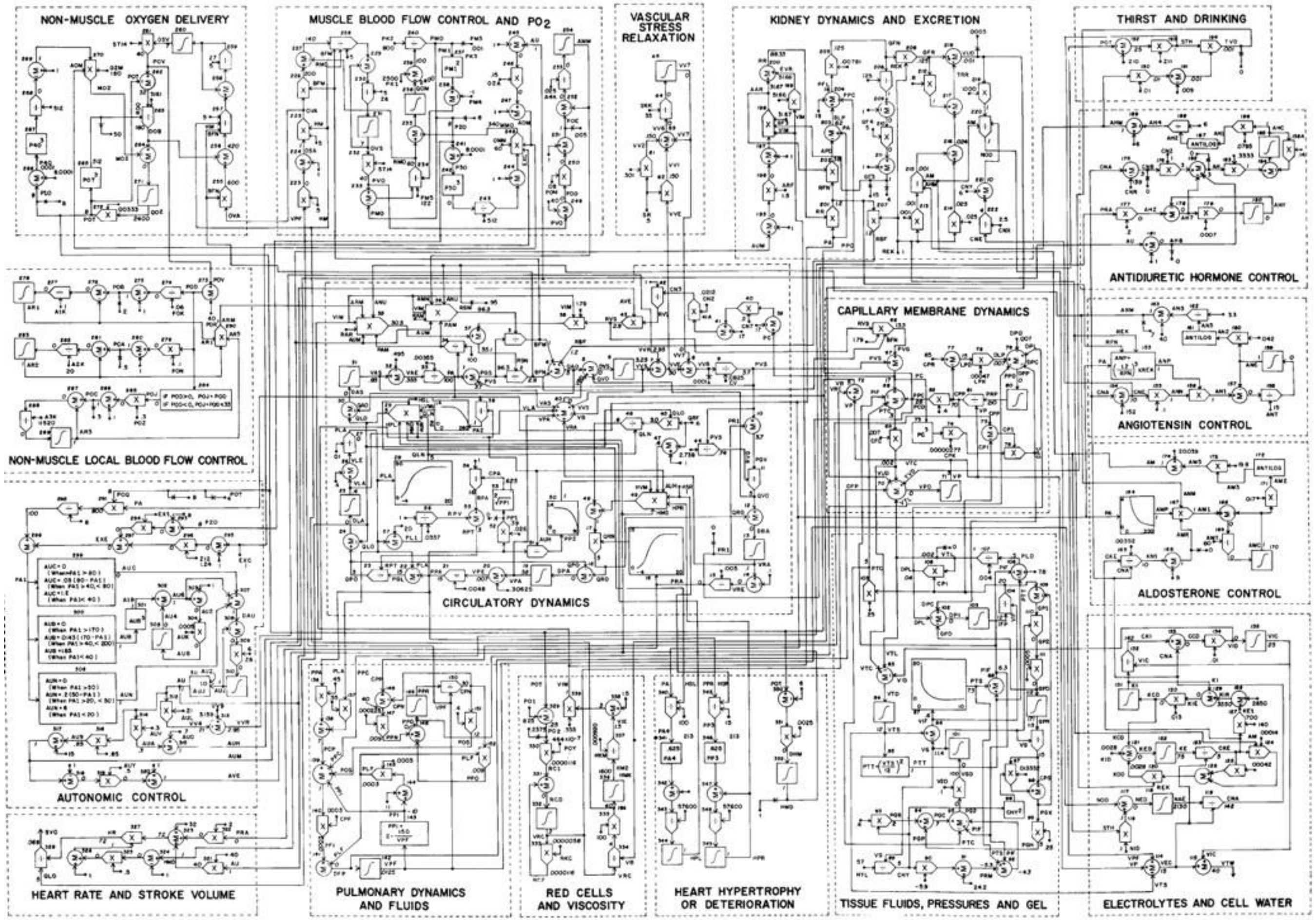
- Resistance = viscous resistance
- Capacitance = vessel compliance
- Inductance = inertial effect
- Complex velocity profiles can be simulated by adding combinations of inductances and resistances in parallel
- Time evolution of pressure and flow in the system
- No spatial discretisation





Jaeger et al. (1959)

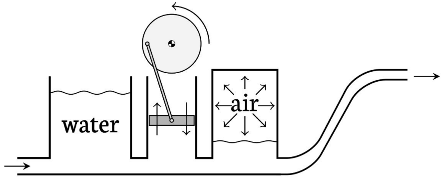
The University Of Sheffield.

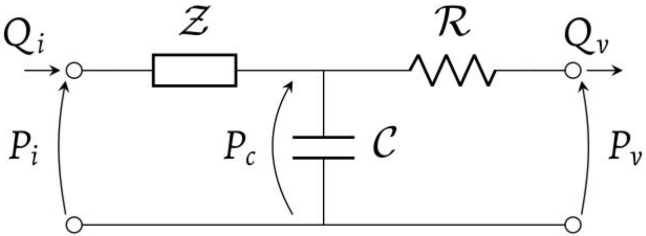


Guyton et al. (1972)

Three element windkessel

- Used to simulate capillary bed
- Replace inductance with resistance (no inertial effects in capillaries)
- Assume fully developed parabolic profile: resistance term from Poiseuille's law
- Assigned constant venous pressure: no pulsatility in capillaries

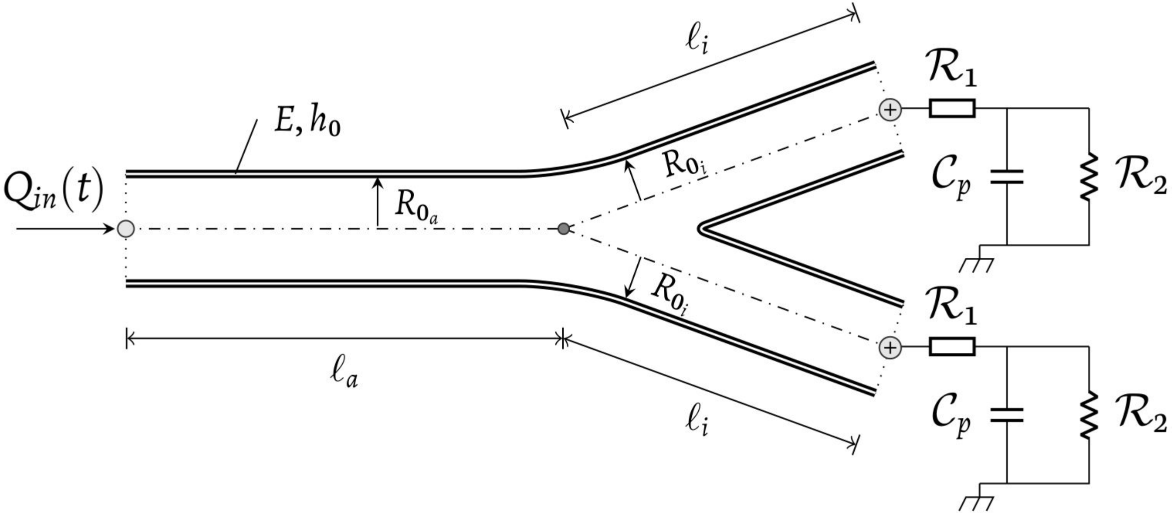




$$Q_i\left(1+\frac{\mathcal{Z}}{\mathcal{R}}\right)+\mathcal{C}\mathcal{Z}\frac{\partial Q_i}{\partial t}=\frac{P_i-P_v}{\mathcal{R}}+\mathcal{C}\frac{\partial P_i}{\partial t}$$

Multiscale vascular model

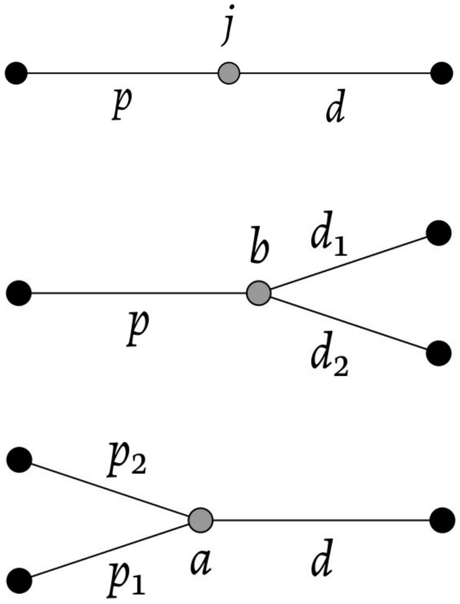
- 1D for long elastic vessels
- 0D for capillaries

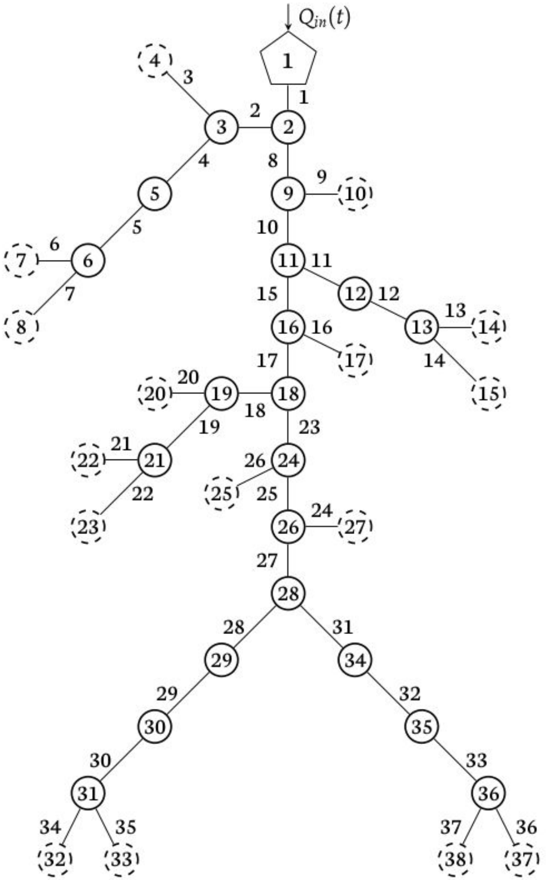


Network construction

- Conjunction
- Bifurcation
- Anastomosis

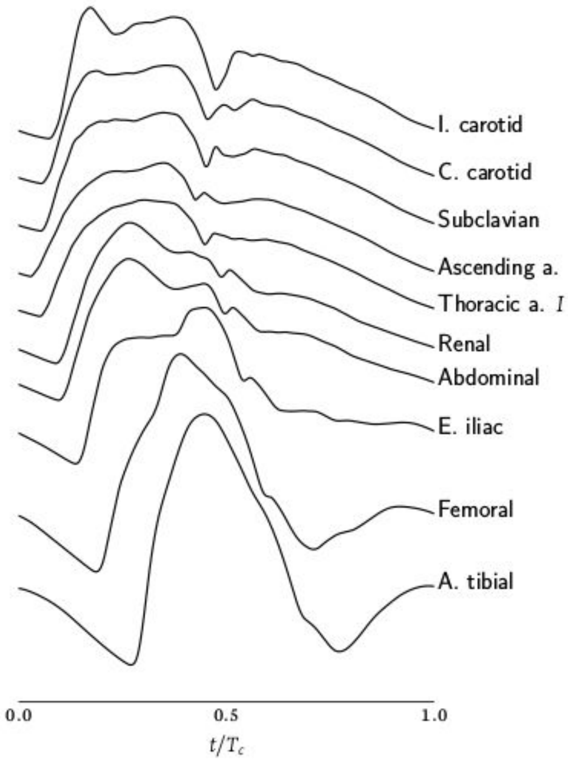
Conservation of mass and static pressure at junction nodes (ideal junctions)

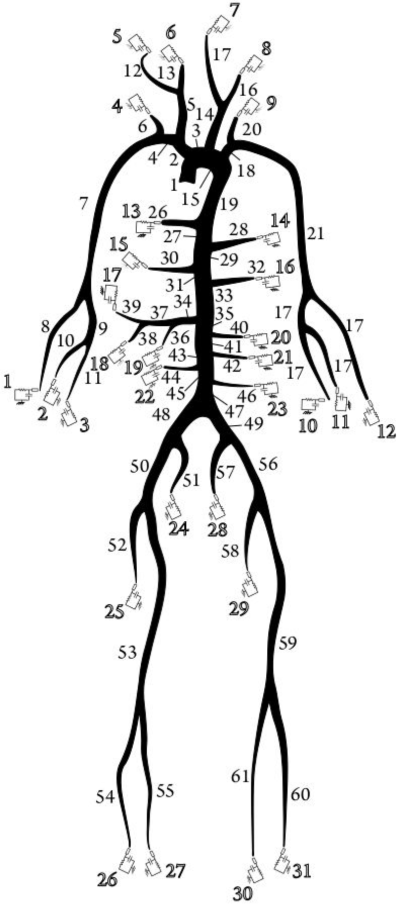




Systemic circulation network

- 61 main elastic arteries
- Three element windkessels at outlets
- Inlet heart function
- Subject-generic mechanical properties from literature
- Gold-standard for 1D model validation

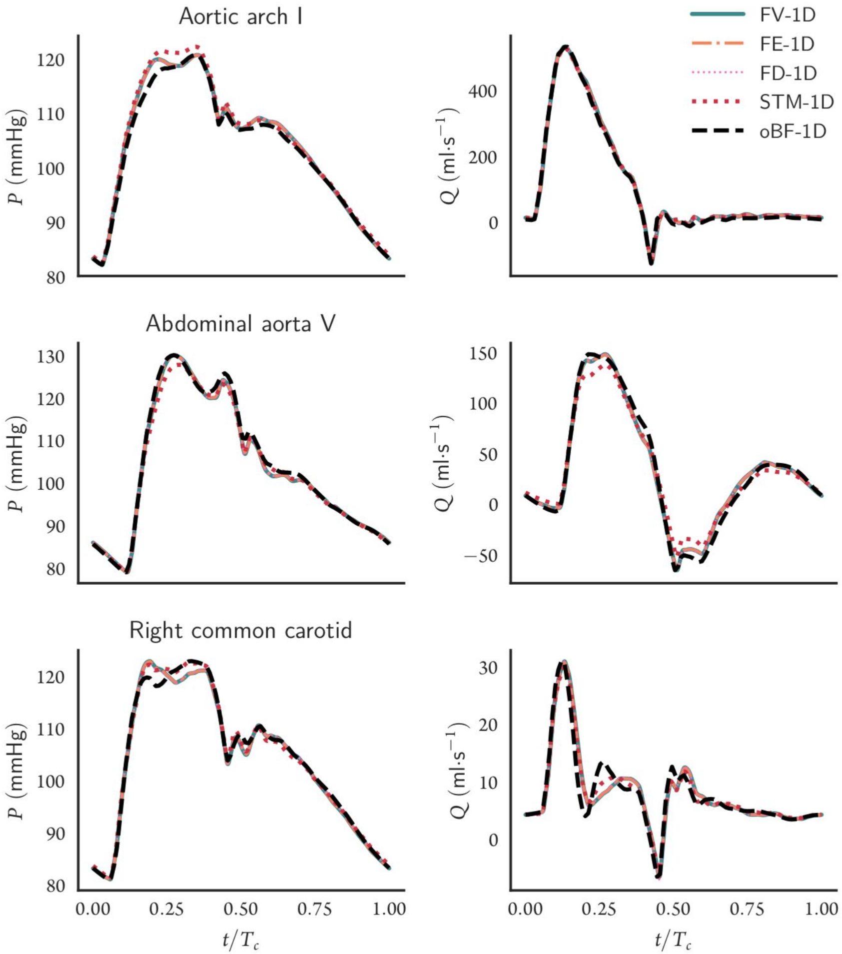




1D solver validation

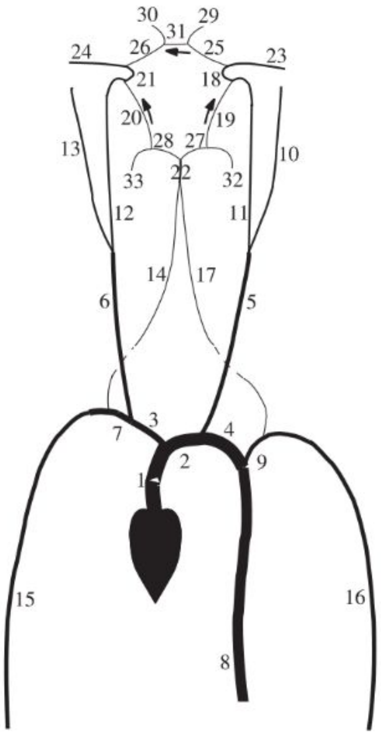
- Comparison between solvers from literature employing different numerical techniques:
  - FV finite volume
  - FE finite element
  - FD finite difference
  - STM simplified trapezoidal method
  - oBF openBF (finite volume)
- openBF

https://github.com/INSIGNEO/openBF

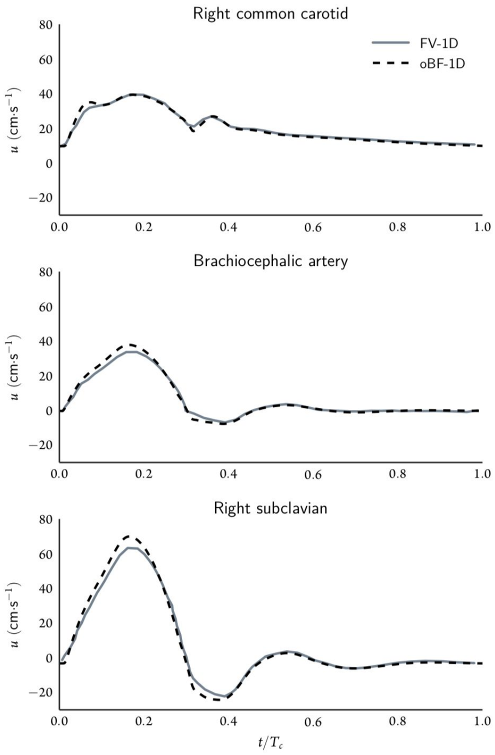


Cerebral circulation

- Complete Circle of Willis network
- Comparison with *in vivo* measurements
- This model can be used for studies on cerebral arteries which are typically very difficult to reach with non invasive techniques

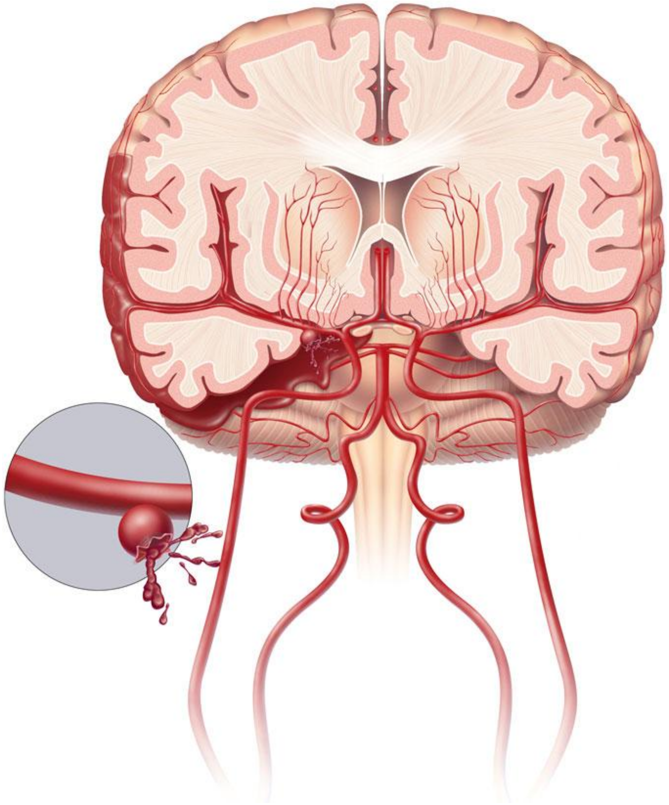


Alastruey et al. (2007)



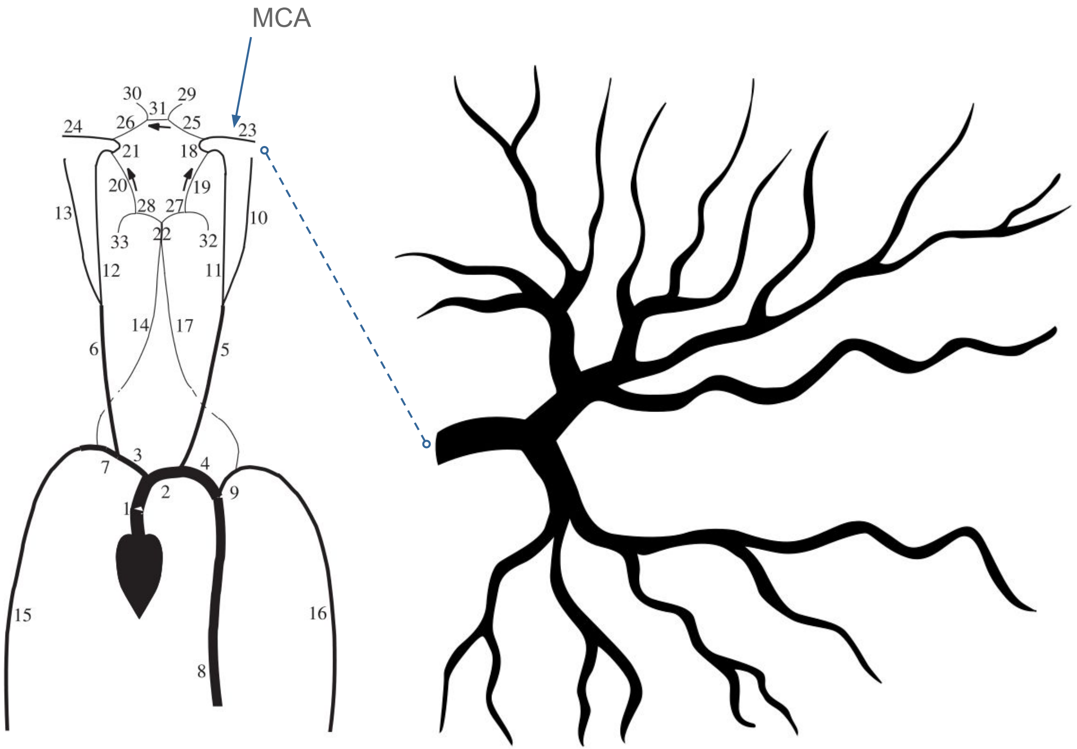
Cerebral vasospasm

- Subarachnoid space haemorrhage following aneurysm rupture.
- Intracranial arteries narrowing.
- Slow onset.
- Peak after one week: ~60% lumen reduction
- May cause cerebral ischemia.
- Current diagnosis tool: velocity measurements at the middle cerebral artery (MCA).



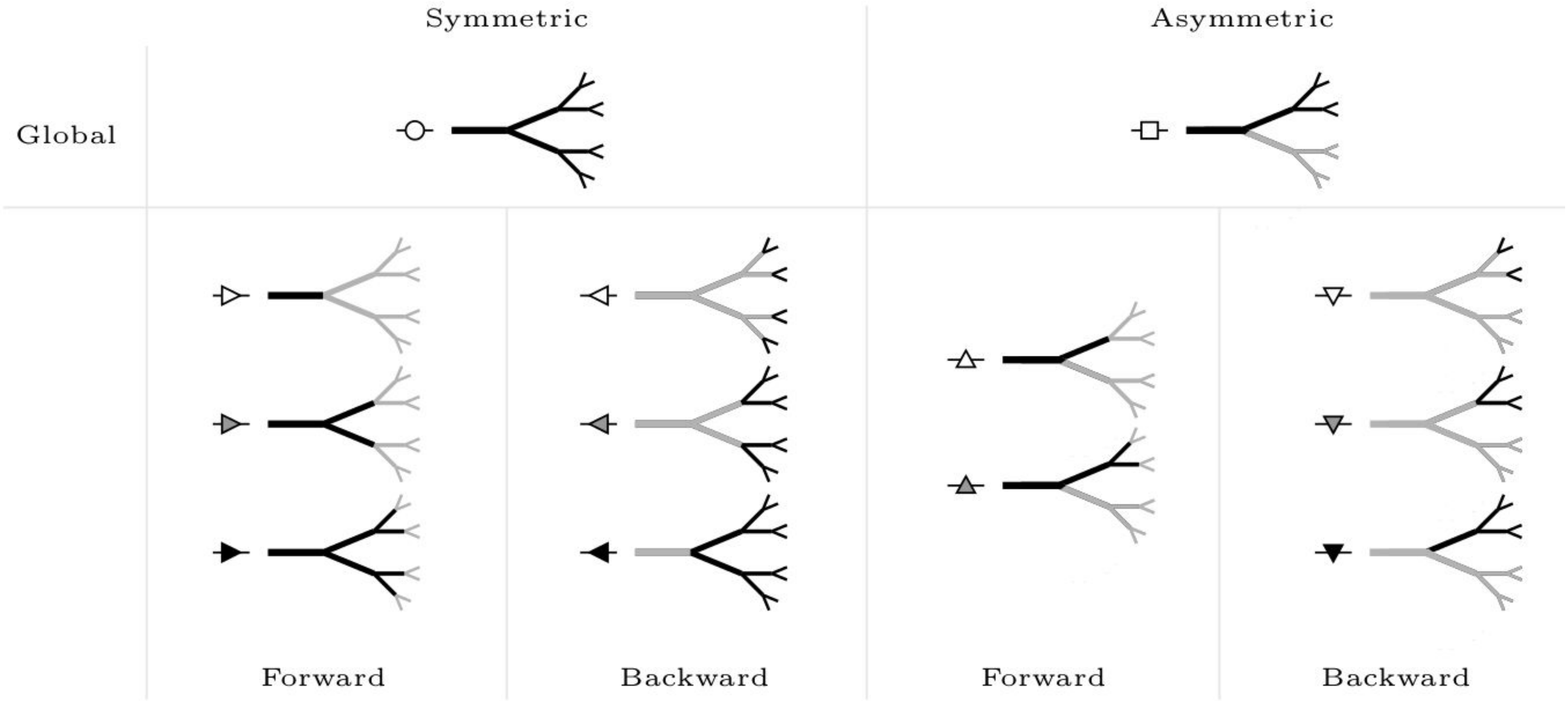
Vasospasm model

- Complete Circle of Willis and post-MCA arterial network (77 segments).
- Vasospasm occurs only in the post-MCA arteries.
- Lumen reduction ranging 0-60%.
- Waveform readings only at MCA mid-point.

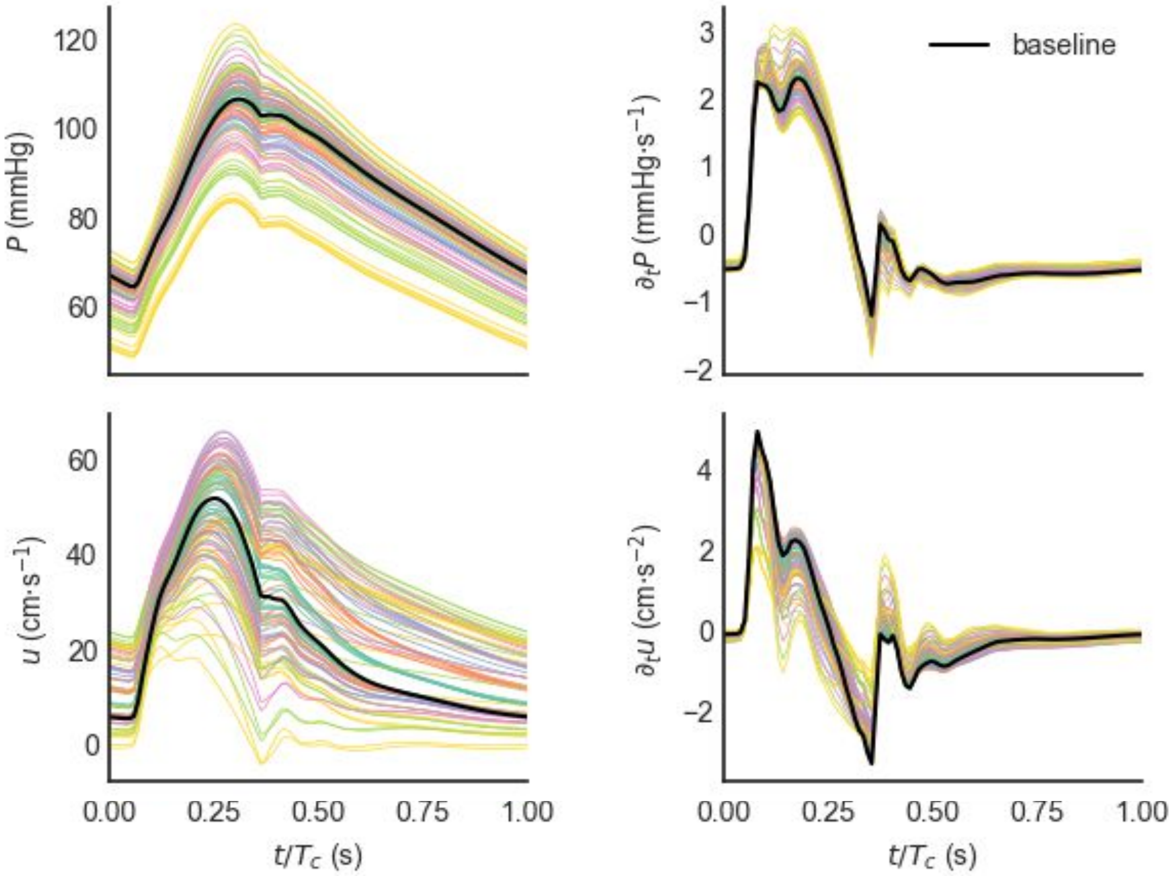


Alastruey et al. (2007) JBM

Vasospasm configurations

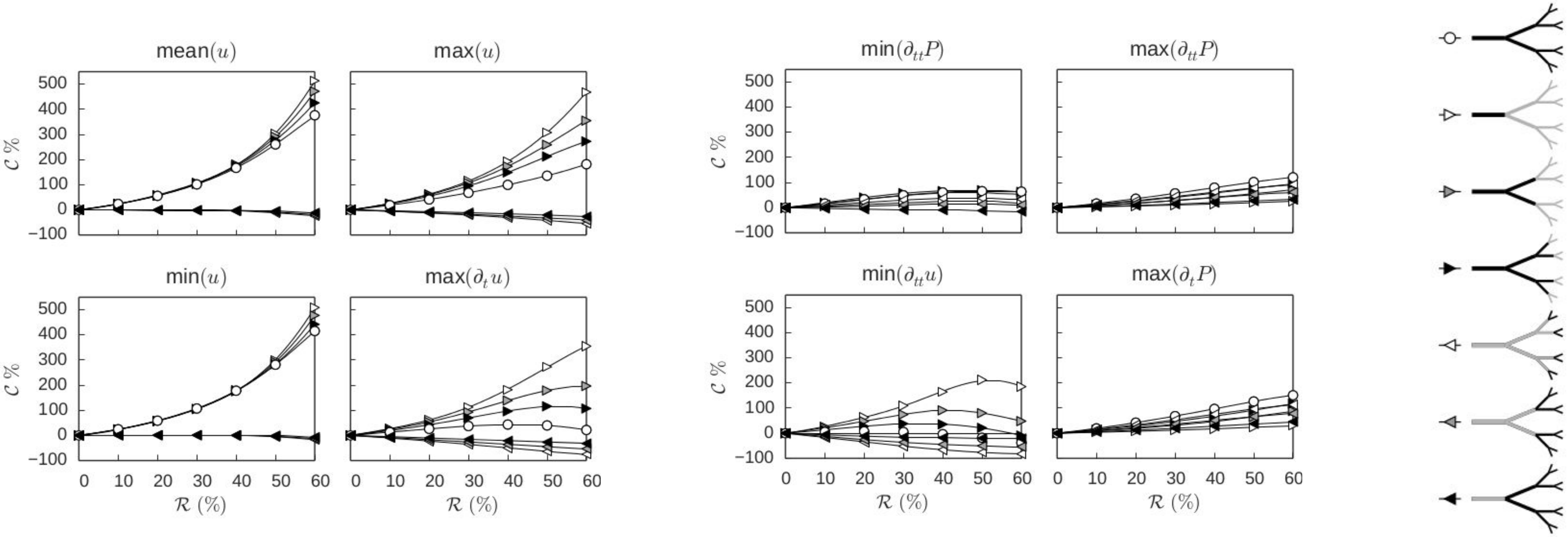


Waveforms



120
100
80
60
P (mmHg)
3
2
1
0
−1
−2
$\partial_t P$ (mmHg·s^{-1})
baseline
60
40
20
0
u (cm·s^{-1})
4
2
0
−2
$\partial_t u$ (cm·s^{-2})
0.00
0.25
0.50
0.75
1.00
t/T_c (s)
0.00
0.25
0.50
0.75
1.00
t/T_c (s)

Biomechanical markers



Summary

- 1D equations comes from the reduction of 3D Navier-Stokes equations
- A parabolic/flat velocity profile
- Linearly elastic constitutive equation
- Time-dependent inlet boundary condition
- Systemic network is built by assembling single vessels
- Junction nodes can be bifurcations, anastomosis, or conjunctions
- Waveform analysis can be applied to detect early onset of complex conditions as the cerebral vasospasm
- Waveforms features carry information about peripheral areas of the circulation that are not directly accessible for measurements.